Constant-Control Rolling Maneuver

Tiberiu Hacker
Institute for Fluid Mechanics and Aerospace Design, Bucharest, Rumania

Practical interest for supplementing stability analysis by domain-of-attraction considerations in studying constant-control maneuvers is pointed out. Implications in cost of computing work of the free parameter choice are estimated. A technique is presented for defining the entire safe range of maneuver in terms of regions of allowable and actually attainable roll rate values and the corresponding aileron inputs. Gravity effects are considered to define peak-value corrections of limits on the steady-state values of system variables and to estimate the maximum allowable duration of a rolling maneuver. An illustration of how a Lyapunov function provides estimate of the domain of attraction is given.

Nomenclature

g	= gravity acceleration
I_x, I_y, I_z	= mass moments of inertia about principal axes
$i_1 = (I_z = I_y)/I_x$, $i_2 = (I_z - I_x)/I_y$, $i_3 = (I_y - I_x)/I_z$	= nondimensional inertia coefficients
l	= rolling moment per I_x
m	= pitching moment per I_y
n	= yaw moment per I_z
n_y, n_z	= side and normal load factors
p, q, r	= scalar components with respect to the principal axes of the aircraft angular velocity
t	= time
u, v, w	= components of V with respect to principal axes
V	= velocity of the aircraft center of mass
y	= side force over aircraft mass and speed
z	= aerodynamic force along z principal axis over mass and speed
α	= incidence
β	= angle of sideslip
$\delta_a, \delta_e, \delta_r$	= deflection angles of ailerons, elevator, and rudder
$\epsilon = g/V$	= specific small parameter
θ	= elevation angle of the x principal axis
ϕ	= angle of bank

Subscripts

a	= maximum allowable absolute value
lim	= not to exceed absolute value
$p, q, r, \alpha, \beta, \dot{\alpha}, \delta_a, \delta_e$	= partial derivatives due to respective quantities [e.g., $y_\beta = \partial y/\partial\beta$, $m_{\dot{\alpha}} = \partial m/\partial(\mathrm{d}\alpha/\mathrm{d}t)$]
s	= steady-state value, effective or potential
1	= coefficient of ϵ in terms of first-order approximation with respect to ϵ

Introduction

As is known,[1,2] in certain aircraft rolling velocity may not decay to zero after the aileron is centered if the aileron-induced motion comes earlier in the neighborhood of a stable[3] autorotational steady state. Then, with zero control deflection, the airplane has a natural tendency to roll with a velocity that oscillates about the stable autorotational value of the roll rate. It also is known that similar pathological motions may be induced by constant controls that differ from zero as well. Steady states corresponding to constant-control inputs were investigated by Rhoads and Schuler[4] and, more recently, by Schy and Hannah.[5] It has been shown that, with the controls at hand, continuous variation of the (constant) value of the control parameters does not necessarily lead to a continuous variation of the resulting steady-state rolling velocity. Thus, in the example computed in Ref. 5 for airplane B, with $\delta_r = 0$ and $\delta_e = 2$ deg, no constant aileron deflection can induce steady rolling with velocities ranging from about 80 to 200 deg/s. Moreover, the continuous variation of the control input may result in a "jump" of the response and cause unwanted, dangerous state excursions. In Ref. 5, a procedure is developed for determining the ranges of potentially critical values of the constant control parameters. The obvious further step is to delimit those subintervals in which the values of these parameters actually are inadmissible or, at least, to to define the highest allowable control input values. The present paper is, for the most part, concerned with this subject.

The phenomena just mentioned are investigated by neglecting weight effects on the response. Such an assumption does not hold any longer when attitude or path-slope considerations are becoming crucial.[3,6] The effects of the gravity terms are considered to estimate peak-value corrections of the limits on the steady-state values of the system variables, particularly the incidence and the sideslip, and to determine some safe bound on the maneuver time.

Assumptions: Equations of Motion

Constant aileron input and aileron plus elevator input maneuvers will be considered. The analysis is, however, identical for any other constant-control combinations, including, e.g., the rudder.

Aerodynamic nonlinearities will be disregarded. Consideration of nonlinear aerodynamics involves no additional difficulty as far as flight dynamics are concerned. Difficulties are encountered only by the aerodynamicists concerned with the estimate of higher-order derivatives.

A few further simplifying assumptions will be made. As usual, we shall assume the c.g. velocity V constant during the entire motion considered. The angle between the x body axis and the direction of the velocity V is assumed small enough at any instant to have $u \cong V$, and the lateral and normal velocity components are sufficiently small for $\alpha \cong w/V$ and $\beta \cong v/V$. Furthermore, the forces and moments acting on the aircraft may be considered independent of the angle of yaw. Lateral and normal forces and the pitching moment induced by aileron deflection also are neglected.

Received July 27, 1977; revision received Jan. 9, 1978.

Index category: Handling Qualities, Stability and Control.

*Principal Scientist; also Head, Flight Dynamics Group. Associate Fellow AIAA.

Under these assumptions, the significant equations of the aircraft motion referred to the principal axes are

$$\dot{\beta} = y_\beta \beta - r + p\alpha + (g/V)\sin\phi \cos\theta \tag{1a}$$

$$\dot{\alpha} = z_\alpha \alpha + q - p\beta + (g/V)\cos\phi \cos\theta + z_c \tag{1b}$$

$$\dot{q} = \bar{m}_\alpha \alpha + \bar{m}_q q + m_r r - m_{\dot{\alpha}} p\beta + i_2 pr + (g/V) m_{\dot{\alpha}} \cos\phi \cos\theta + m_c \tag{1c}$$

$$\dot{r} = n_\beta \beta + n_q q + n_r r + n_p p - i_3 pq + n_c \tag{1d}$$

$$\dot{p} = l_\beta \beta + l_q q + l_r r + l_p p - i_1 qr + l_c \tag{1e}$$

$$\dot{\phi} = p + q \sin\phi \tan\theta + r \cos\phi \tan\theta \tag{1f}$$

$$\dot{\theta} = q \cos\phi - r \sin\phi \tag{1g}$$

where

$$\bar{m}_\alpha = m_\alpha + z_\alpha m_{\dot{\alpha}}, \quad \bar{m}_q = m_q + m_{\dot{\alpha}}, \quad z_c = z_{\delta_e}\delta_e,$$
$$m_c = m_{\delta_e}\delta_e, \quad n_c = n_{\delta_a}\delta_a, \quad \text{and} \quad l_c = l_{\delta_a}\delta_a$$

Except for the sections where gravity effects also are considered, attention is focused on the behavior of the dynamic variables β, α, q, r, and p, and accordingly only Eqs. (1a-1e), with the terms that have g/V as a factor suppressed, are considered.

The flight régime at which the maneuver is initiated is referred to as the initial state throughout. Specifically, it is characterized by low or zero values of the variables of the system [e.g., $(0, \alpha_0, 0,0,0,0,\alpha_0)$, where $\alpha_0 = -\epsilon/z_\alpha$, $\epsilon = g/V$].

Stability and Domain of Attraction

To study constant-control rolling maneuvers, the target roll rate[4] or the aileron angle input[5] is optionally used in literature as the free parameter, occasionally along with the elevator and/or rudder input. In the author's opinion both choices are equivalent inasmuch as they involve the use of similar though not equally costly techniques and, in effect, yield indentical results. A comparative estimate of cost implications will be considered briefly in the next section.

Whatever the choice of the free parameter, analysis would start with seeking the stable singular points of system (1a-1e) with $\epsilon = 0$, corresponding to the given constant value of the independent parameter. Singular points are obtained as solutions of the system of nonlinear algebraic equations that result when the time derivatives of the variables are set equal to zero. The next step is to check whether the stable singular points are attractive or not. That is to say, it is necessary to establish which, if any, of the potential steady states actually will take place.

A constant control effectively achieves a certain steady state described by a stable singular point of system (1a-1e) with $\epsilon = 0$ only if the initial state lies in the domain of attraction of this steady state. It should be recalled that the domain of attraction of a singular point is the set of all initial states that eventually approach that point. When this is the case, the steady state will be referred to as "effective attractor" or "attractor" (with respect to the given initial state). The attractor is safe if none of the coordinates of the singular point (incidence, sideslip, etc.) exceeds some given allowable bound, and the applied control is safe if and only if it admits of a safe attractor. This means that the input value is critical when it admits either an attractor whose coordinates are in some dangerous region (e.g., where high load factors are involved) or no singular point that would attract trajectories issuing from the region of interest of the state space. Accordingly, dangerous state excursions are implied by the absence of an attractive safe singular point rather than by the mere existence of stable singular points situated in some critical region.

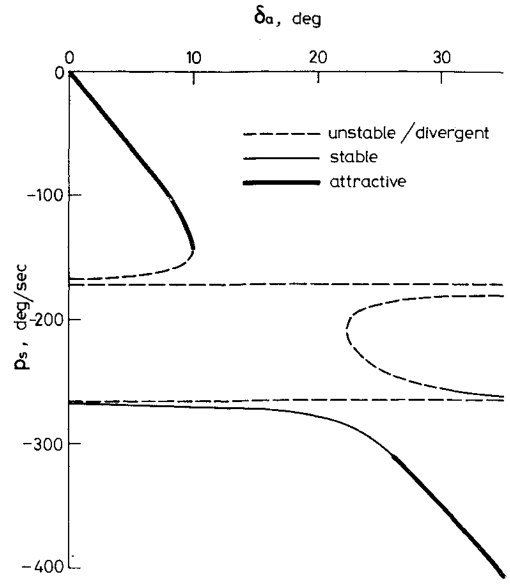

Fig. 1 Negative p_s coordinates of singular points corresponding to constant aileron inputs for $\delta_e = 2$ deg.

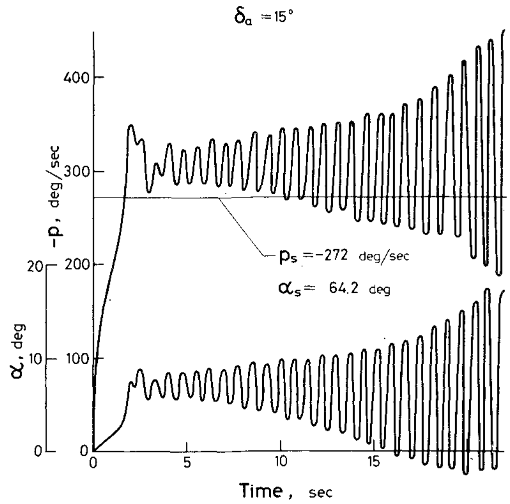

Fig. 2 Time histories of roll rate and incidence responses to 15-deg aileron angle and 2-deg elevator angle inputs.

In Fig. 1, the curves referred to in Ref. 5 as pseudo-steady state (PSS) solutions are plotted for our numerical example (see Appendix A) for $\delta_e = 2$ deg, with supplementary emphasis on the attractiveness of the stable branches. Only negative p_s solutions are plotted for simplicity. As can be seen, no singular point is attractive for zero initial conditions (or for any in the neighborhood of the origin) for aileron deflections ranging from 10 to 26 deg. There is, however, a stable branch in this interval, namely, that which becomes attractive further on when the aileron angles exceed 26 deg. The time histories for zero initial conditions of the roll rate and the incidence responses to 15- and 30.76-deg aileron inputs are plotted in Figs. 2 and 3. We have a stable singular point belonging to the same branch of the PSS diagram for each of these input

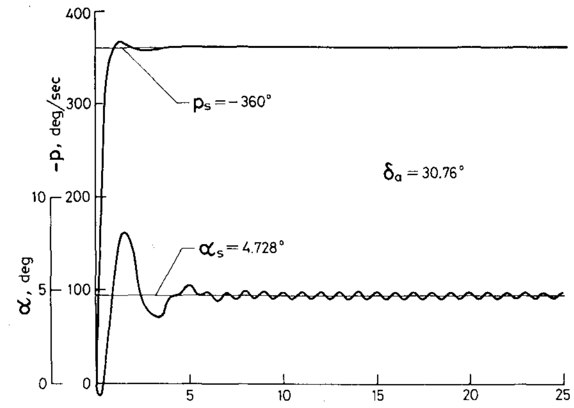

Fig. 3 Same as Fig. 2, but $\delta_a = 30.76$ deg.

values. The origin, however, lies in the domain of attraction of the singular point with $p_s = -360$ deg/s and $\alpha_s = 4.73$ deg, corresponding to $\delta_a = 30.76$ deg, and not in that of the singular point with $p_s = -272$ deg/s and $\alpha_s = 64.2$ deg, corresponding to $\delta_a = 15$ deg. The input $\delta_a = 15$ deg admits of no singular point that attracts solutions issuing from the neighborhood of the origin, although there exists a stable singular point. It should be noted that the unstable oscillations for $\delta_a = 15$ deg cannot be ascribed to gravity effects. These effects only cause the incidence amplitudes to increase by up to 15% in the first 20 s.

Discussion of Free Parameter Choice

When roll rate is made the free parameter (along with δ_e), determination of the singular points involves the solution of a second-degree equation and a set of four linear equations. Accordingly, two aileron inputs and two singular points result for each roll rate value. Only one of these is instrumental, as a rule. The alternate solution is usually both unstable and unnatural as regards magnitudes and/or signs (see "Numerical Example," point 3).

When the aileron input is used as the independent parameter (along with δ_e), a ninth-degree equation and a set of four linear equations must be solved. Nine sets of roots result, several of which may be real and stable and have realistic values. For instance, for $\delta_a = 17$ deg, $\delta_e = -2$ deg we have two stable singular points with negative p_s, namely, $p_s = -176$ deg/s ($\beta_s = -8.28$ deg) and $p_s = -256$ deg/s ($\beta_s = 20.3$ deg); for $\delta_a = 21$ deg, $\delta_e = 0$, again we have two stable singular points with negative p_s, viz., $p_s = -186$ deg/s ($\beta_s = -22.7$ deg) and $p_s = -269$ deg/s ($\beta_s = 18.7$ deg), and so on. See also the PSS curves in Ref. 5.

To define the safe range of aileron inputs or predict the imminence of critical state excursions, a greater number of singular points must be tested when the aileron input instead of the target roll rate is made the free parameter. This affects efficiency of the method, since testing is often a costly process. The least burdensome way of testing is the direct estimate of the domain of attraction of the stable singular points. Unfortunately, however, no comprehensive working technique is available for estimating the domain of attraction. Lyapunov's direct method can provide an estimate if an appropriate Lyapunov function is found.[7-9] An illustration of how a Lyapunov function provides an estimate is given in Appendix B. A quadratic form connected with the linear variational system is used as a Lyapunov function with a mere illustrative purpose. It yields valid conclusions for small aileron inputs. Any effort to establish a closer connection between the particular form of the system including the nonlinear terms and the way of constructing a specific Lyapunov function, in order to obtain a better estimate, will be welcome. When no effective technique is available for domain-of-attraction estimates, numerical integration is the only alternative. An algorithm will be sketched further for constant-control rolling maneuver analysis, with a view to reducing computing work as far as possible by a parsimonious preselection of singular points to be tested.

Definition of Safe Maneuver Range

The algorithm includes the following:

1) Flight conditions are chosen. A set of typical and/or potentially critical flight conditions will be considered.

2) Target steady-state roll rates are chosen. Computations are performed for parameter p_s varying from zero up to that corresponding to the airplane's maximum ideal roll rate capability [defined by the expression $(l_{\delta_a}/l_p) \times$ full aileron deflection] by sufficiently small increments.

3) The four remaining coordinates of the related singular points $(\beta_s, \alpha_s, q_s, r_s)$ and the aileron deflection δ_a are determined. They are defined by

$$\begin{bmatrix} y_\beta & p_s & 0 & -1 \\ -p_s & z_\alpha & 1 & 0 \\ -m_{\dot\alpha} p_s & \bar m_\alpha & \bar m_q & m_r + i_2 p_s \\ n_\beta & 0 & n_q - i_3 p_s & n_r \end{bmatrix} \begin{bmatrix} \beta_s \\ \alpha_s \\ q_s \\ r_s \end{bmatrix}$$

$$= -\delta_a \begin{bmatrix} 0 \\ 0 \\ 0 \\ n_{\delta_a} \end{bmatrix} - \delta_e \begin{bmatrix} 0 \\ z_{\delta_e} \\ m_{\delta_e} \\ 0 \end{bmatrix} - p_s \begin{bmatrix} 0 \\ 0 \\ 0 \\ n_p \end{bmatrix} \tag{2a}$$

$$l_\beta \beta_s + l_q q_s + l_r r_s - i_1 q_s r_s = -l_p p_s - l_{\delta_a} \delta_a \tag{2b}$$

β_s, α_s, q_s, and r_s are obtained as linear functions of δ_a (and the parameters p_s and δ_e) from Eqs. (2a). Then Eq. (2b) becomes a second-degree equation in δ_a. Two sets of values for β_s, α_s, q_s, r_s, and δ_a are obtained for some given values of δ_e and p_s (out of which only one turns to account, as a rule).

4) Each value is inspected to ascertain whether it is situated in the admissible region. (For the definition of the admissible region, see the following section.) If it is not and/or $\delta_a p_s \geq 0$, the process is discontinued for the given singular point.

5) If the values of β_s, α_s, q_s, r_s, and δ_a are acceptable, we shall check whether the initial state lies in the domain of attraction of the singular point. If no technique is available to estimate the domain of attraction, the verification is done by integrating system (1) for proper initial conditions over a long enough (say 10-30 s) interval. If it results that the steady state corresponding to singular point (β_s, α_s, q_s, r_s, p_s) actually is approached, then we go to the next point.

6) The aileron deflection δ_a is plotted against p_s.

Effect of Gravity Terms

Correction for Admissible Bounds on the Steady-State Values of Dynamic Variables

The technique just outlined requires the knowledge of some safe bounds on the steady-state values of β, α, q, r (see point 4). Now weight effect may induce rather large oscillatory components of the response involving high peak values. Therefore, corrections allowing for this effect should be introduced. To this end, the solution of Eqs. (1) is sought in the form[3,6]

$$\beta = \beta_s + \epsilon\beta_1, \; \alpha = \alpha_s + \epsilon\alpha_1, \; q = q_s + \epsilon q_1,$$
$$r = r_s + \epsilon r_1, p = p_s + \epsilon p_1, \; \phi = \phi_s + \epsilon\phi_1, \; \theta = \theta_s + \epsilon\theta_1$$

where $\epsilon = g/V$, and β_s, α_s, q_s, etc., are the respective steady-state values. The bound imposed upon θ and the effective values of q and r are small enough to have $\tan\theta \cong \theta$, $q/p \ll 1$, $r/p \ll 1$, and $q\theta$ and $r\theta$ negligible as compared to p. Then it can be seen that $\phi_s = p_s t$ for $\phi_s(0) = 0$ (see next subsection). Accordingly,

$$\dot{x}_I = Ax_I + h(t) \tag{3}$$

where $x_I = (\beta_I, \alpha_I, q_I, r_I, p_I)^T$,

$$A = \begin{bmatrix} y_\beta & p_s & 0 & -1 & \alpha_s \\ -p_s & z_\alpha & 1 & 0 & -\beta_s \\ -m_{\dot{\alpha}} p_s & \bar{m}_\alpha & \bar{m}_q & m_r + i_2 p_s & -m_{\dot{\alpha}}\beta_s + i_2 r_s \\ n_\beta & 0 & n_q - i_3 p_s & n_r & n_p - i_3 q_s \\ l_\beta & 0 & l_q - i_I r_s & l_r - i_I q_s & l_p \end{bmatrix}$$

$$h(t) = \cos\theta_s\,[\sin p_s t,\ \cos p_s t, m_{\dot{\alpha}} \cos p_s t, 0, 0]^T$$
$$\cong [\sin p_s t,\ \cos p_s t, m_{\dot{\alpha}} \cos p_s t, 0, 0]^T$$

This system admits a periodic solution of the form $x_I = x_I' \sin p_s t + x_I'' \cos p_s t$, where the components β_I', α_I',..., and β_I'', α_I'',... of x_I' and x_I'', respectively, are defined by the following set of 10 algebraic equations:

$$Ax_I' + p_s I x_I'' = -\cos\theta_s\,[1,0,0,0,0]^T \cong -[1,0,0,0,0]^T$$
$$-p_s I x_I' + Ax_I'' = -\cos\theta_s\,[0,1,m_{\dot{\alpha}},0,0]^T \cong -[0,1,m_{\dot{\alpha}},0,0]^T$$

where I stands for the fifth-order unit matrix. Let $\tilde{\beta}_I'$, $\tilde{\alpha}_I'$,..., $\tilde{\beta}_I''$, $\tilde{\alpha}_I''$,... be the solution, and denote $\Delta\beta = \epsilon(\tilde{\beta}_I'^2 + \tilde{\beta}_I''^2)^{1/2}$, $\Delta\alpha = (\tilde{\alpha}_I'^2 + \tilde{\alpha}_I''^2)^{1/2}$,.... Then, if β_a, α_a, q_a, r_a are the highest allowable absolute values of the corresponding variables, defined by considerations of aerodynamics (stalling) or inertia (loading of the structure or the crew), we shall require that

$$|\beta_s| \leq |\beta_a - \Delta\beta| = \beta_{s_{\lim}},\ |\alpha_s| \leq |\alpha_a - \Delta\alpha| = \alpha_{s_{\lim}}$$

and so on, in addition to the obvious limitation imposed upon δ_a. The importance of the effect of the gravity terms critically depends upon the level of the highest admissible values, as illustrated in the numerical example in the following section.

Maneuver Time Limitation

It has been shown[6] that the weight effect is essential as regards the angular attitude of the airplane and the flight path angle. Because of the gravity effect, the flight path can only be maintained (approximately) level by using a certain oscillatory control input in nonresonant situations and cannot be enforced near resonance. The model of steady rolling has been considered in Ref. 6. Even this ideal maneuver is exponentially (though slightly) unstable with respect to attitude when control is performed with realistic accuracy. As will be shown, constant-control maneuver is always divergent as regards attitude: weight effect induces a rather heavy attitude divergence, which is described by the presence in the expression of $\epsilon\theta_I$ of a secular term induced by the gravity terms in the equations. With the constant rolling constraint, secular terms can be avoided[3,6] by suitable choosing the control functions, which is not the case of constant controls. If control is chosen in accordance with the procedure presented in the preceding section (which insures safe bounds on incidence, sideslip, and angular velocities), the maneuver time is limited by attitude divergence only. An estimate of the allowable limit on the duration of the maneuver is given in the following.

Yaw and pitch rates are small as a rule. Their order of magnitude actually can be assumed equal to $O(\epsilon)$. Denote $\bar{q}_s = q_s/\epsilon$, $\bar{r}_s = r_s/\epsilon$. Then, according to Eqs. (1f) and (1g), we have

$$\dot{\phi}_s + \epsilon\dot{\phi}_I = p_s + \epsilon[p_I + (\bar{q}_s + q_I)\theta_s \sin\phi_s + (\bar{r}_s + r_I)\theta_s \cos\phi_s]$$
$$\dot{\theta}_s + \epsilon\dot{\theta}_I = \epsilon[(\bar{q}_s + q_I)\cos\phi_s - (\bar{r}_s + r_I)\sin\phi_s]$$

where $\phi_s = p_s t$ for $\phi_s(0) = 0$, $\theta_s = \theta_s(0) = \text{const}$, and

$$\theta_I = \theta_I(0) + [(q_I'' - r_I')/2]t + [-\bar{r}_s + (q_I' - r_I'')/4]/p_s$$
$$+ \{\bar{q}_s \sin p_s t + \bar{r}_s \cos p_s t - [(q_I' - r_I'')/4]\cos 2p_s t$$
$$+ [(q_I'' + r_I')/4]\sin 2p_s t\}/p_s \tag{4}$$

As is seen, θ contains the secular term $\epsilon t(q_I'' - r_I')/2$. Permissible maneuver time may be inferred from relation (4) for small p_s values. Let θ_a be the allowable upper limit on $|\theta|$. Then the maximum allowable maneuver time t_{RM} can be estimated [e.g., graphically; see Fig. 4, 1) according to formula (4) for $p_s = 0.5$ rad/s, and 2) as obtained by numerical integration of Eqs. (1) for $\delta_a = -2.37$ deg, $\delta_e = 0$, and zero initial conditions; permissible maneuver time is estimated for $\theta_a = 20$ deg] from $|\theta_s + \epsilon\theta_I(t_{RM})| < \theta_a$. The time history of θ, as obtained from relation (4) for $p_s = 0.5$ rad/s and by numerical integration of Eqs. (1) for the corresponding values of the control inputs ($\delta_a = -2.37$ deg, $\delta_e = 0$), and zero initial conditions are compared in Fig. 4. For $\theta_a = 20$ deg, the permissible maneuver time is equal to 9.8 s according to Eq. (4) and to 10.8 s from the simulated time history of θ.

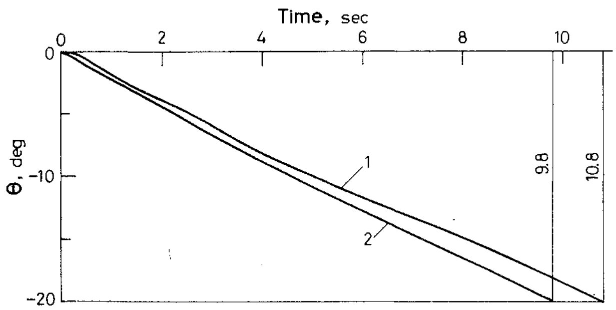

Fig. 4 Permissible maneuver time estimate.

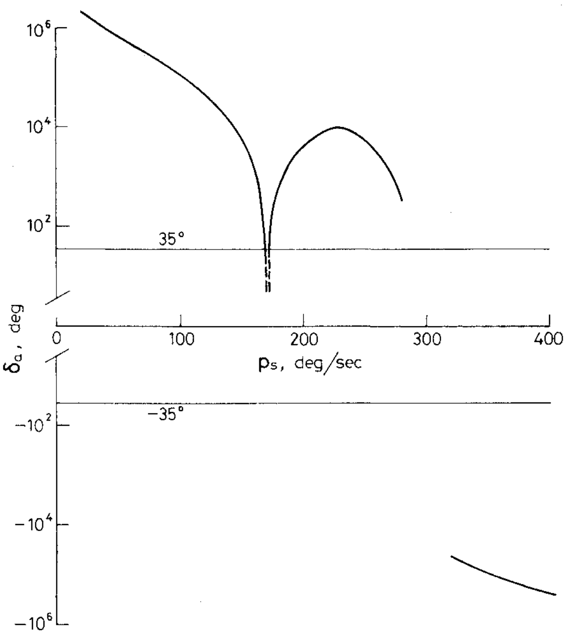

Fig. 5 The secondary branch of the solution in δ_a of Eqs. (2).

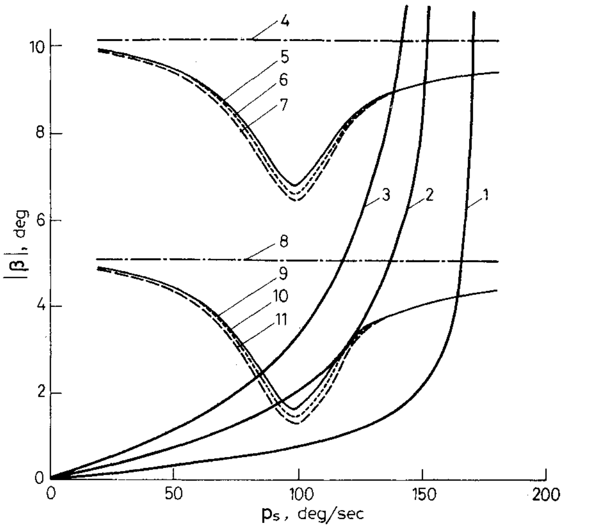

Fig. 6 Steady-state sideslip angle vs target constant rate of roll as compared to its limit.

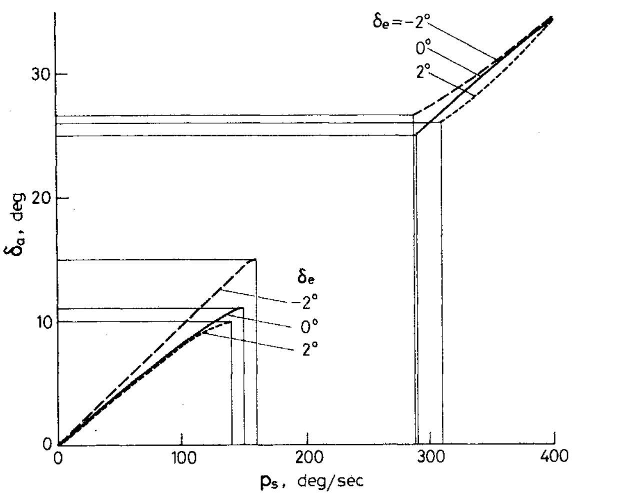

Fig. 7 Achievable steady-state values of roll rate and corresponding constant aileron inputs belonging to the main solution of Eqs. (2).

Numerical Example from Appendix A Data

1) Computations are illustrative and carried out only for one flight régime ($M = 0.9$, altitude = 20,000 ft) and three elevator angles (− 2, 0, and 2 deg).

2) Roll rate values ranging from zero to 400 deg/s have been considered.

3) One of the two branches (referred to as secondary) of roots in δ_a of Eq. (2b) plotted against p_s in Fig. 5 results practically independent of δ_e. It is situated outside the range of the airplane's aileron angles, except for $169 < p_s < 172$. In this interval, however, $\delta_a p_s > 0$. In fact, the product $\delta_a p_s$ is positive for all p_s values up to about 300 deg/s. Moreover, the roots on this branch are, for the most part, too large to have any significance. Thus the absolute values of the negative δ_a roots (for $p_s > 300$ deg/s) exceed 10^4. Hence only the main branch will be used in calculations.

4) The highest permissible absolute values of β_s and α_s have been taken as follows: $\beta_a = -\epsilon n_{y_a}/y_\beta$, $\alpha_a = \min(0.9\ \alpha_{\text{stall}}, -\epsilon n_{z_a}/z_\alpha)$. The maxima of the corrections $\Delta\beta$, allowing for the gravity terms, resulted as shown in Table 1.

5) No constant aileron deflection can induce steady rolling with velocities ranging from 160 to 288 deg/s for $\delta_e = -2$ deg, from 150 to 290 deg/s for $\delta_e = 0$, and from 143 to 310 deg/s for $\delta_e = 2$ deg. For $n_{y_a} = 1$, the allowable limit value $|\beta_a - \Delta\beta|$ of β_s is not exceeded for p_s less than 160, 150, and 143, respectively. For $n_{y_a} = 0.5$, $|\beta_s| > |\beta_a - \Delta\beta|$ for $p_s = 92$ deg/s when $\delta_e = 0$, and for $p_s = 83$ deg/s when $\delta_e = 2$ deg (see Fig. 6). For the upper range of the steady-state roll rate values, β_s, α_s, q_s, and r_s do not exceed their admissible limits. The following critical ranges of the constant δ_a values result: for $n_{y_a} = 1.0$, from 13 to 26.6 deg for $\delta_e = -2$ deg, from 11.04 to 25 deg for $\delta_e = 0$, and from 10 to 26 deg for $\delta_e = 2$ deg; for $n_{y_a} = 0.5$, from 13 to 26.6 deg for $\delta_e = -2$ deg, from 7.4 to 25 deg for $\delta_e = 0$, and from 6.7 to 26 deg for $\delta_e = 2$ deg.

6) Conclusions are summarized in Figs. 7 and 8. Diagrams such as in Fig. 7, with its curves restricted according to point 4 ($|x_{i_s}| \le |x_{i_a} - \Delta x_i|$, Δx_i due to weight effect), are the only ones to be calculated and plotted in applications.

Table 1 Maxima of corrections $\Delta\beta$

δ_e, deg	−2	0	2
$\Delta\beta$, deg	3.32	3.53	3.67
$100\Delta\beta/\beta_a$, %			
For $n_{y_a} = 1.0$	32.7	34.8	36.1
For $n_{y_a} = 0.5$	65.5	69.6	72.2
Resulting[a] $-\Delta p_{s_{\lim}}$, deg/s			
For $n_{y_a} = 1.0$	1	1	3
For $n_{y_a} = 0.5$	2.5	45	35

[a] See Fig. 6, where curves 1-3 represent the resulting steady-state values, and 5-7 for $n_{y_a} = 1$ and 9-11 for $n_{y_a} = 0.5$ represent the limits allowing for gravity effects of β_s for the elevator angle equal to −2, 0, 2 deg, respectively; lines 4 and 8 indicate the maximum allowable absolute value of β for n_{y_a} equal to 1 and 0.5, respectively.

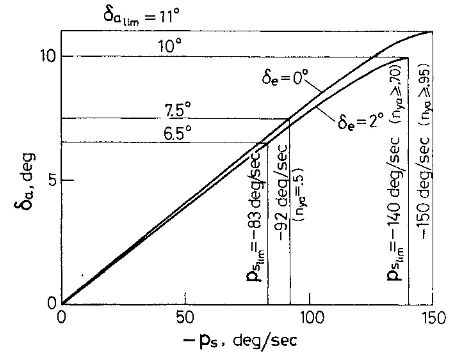

Fig. 8 Admissible roll rates in the range up to 150 deg/s vs the corresponding aileron inputs, allowing for gravity effect.

Appendix A: Characteristics of the Example Airplane and Flight Conditions[3,4]

The airplane and flight condition considered in Refs. 3 and 6 are considered, with elevator deflection angles equal to −2, 0, and 2 deg. Other data are as follows:

Mach number = 0.9 Altitude = 20,000 ft

$i_1 = 0.727,\quad i_2 = 0.949,\quad i_3 = 0.716$

$$y_\beta = -0.196\ \text{s}^{-1},\quad z_\alpha = -1.329\ \text{s}^{-1},\quad z_{\delta_e} = -0.168\ \text{s}^{-1}$$

$$m_\alpha = -23.18\ \text{s}^{-2}\quad n_\beta = 5.670\ \text{s}^{-2}\quad l_\beta = -9.990\ \text{s}^{-2}$$

$$m_q = -0.814\ \text{s}^{-1}\quad n_q = 0.0\quad l_q = 0.107\ \text{s}^{-1}$$

$$m_r = -0.254\ \text{s}^{-1}\quad n_r = -0.235\ \text{s}^{-1}\quad l_r = 0.126\ \text{s}^{-1}$$

$$m_{\dot\alpha} = -0.173\ \text{s}^{-1}\quad n_p = 0.002\ \text{s}^{-1}\quad l_p = -3.933\ \text{s}^{-1}$$

$$m_{\delta_e} = -28.18\ \text{s}^{-2}\quad n_{\delta_a} = -0.921\ \text{s}^{-2}\quad l_{\delta_a} = -45.83\ \text{s}^{-2}$$

Appendix B: Domain of Attraction Estimate: An Illustration

Denote the coordinates of a singular point of system (1a-1e) with $\epsilon = 0$ by β_s, α_s, q_s, r_s, p_s and $x_1 = \beta - \beta_s$,

$x_2 = \alpha - \alpha_s$, $x = q - q_s$, $x_4 = r - r_s$, $x_5 = p - p_s$. Then the first five equations of system (1), with the gravity terms canceled, become

$$\dot{x} = Ax + k(x) \qquad \text{(B1)}$$

where matrix A is as in Eq. (3), and the vector function $k(x)$ of the nonlinear terms is

$$[x_2 x_5, -x_1 x_5, -m_{\dot{\alpha}} x_1 x_5 + i_2 x_4 x_5, -i_3 x_3 x_5, -i_1 x_3 x_4]^T$$

Stability is inferred from the variational system $\dot{x} = Ax$, and the domain of attraction of system (B1) is the set of all initial points x_0 of solutions that eventually approach the origin.

Let $\mathcal{V}(x)$ be a Lyapunov function for system (B1). A region defined by the inequality $\mathcal{V}(x) \leq c = \text{const}$ is an estimate of the complete domain of attraction if $\dot{\mathcal{V}}$ along the integral curves of Eq. (B1) is less than zero within this region. The estimate is better the higher the level c. [The region defined by the surface $\mathcal{V}(x) = c$ tangent to the $\dot{\mathcal{V}} = 0$ surface, and which does not intersect it, provides the best estimate for a given Lyapunov function $\mathcal{V}$.]

We shall use a Lyapunov function associated with the linear system $\dot{x} = Ax$. It will, as usual, be determined as a quadratic form $\mathcal{V} = \Sigma_{i,j} v_{ij} x_i x_j$, with $v_{ij} = v_{ji}$, from the equation $A^T V + VA = D$, where V is the matrix of the quadratic form $\mathcal{V}$, and D is a given negative definite symmetrical matrix. We choose $D = \text{diag}(y_\beta, z_\alpha, \bar{m}_q, n_r, l_p)$. Only those singular points will be considered for which all eigenvalues of V are positive (the stable singular points).

We now shall delimit a region in the x space defined by the inequality $\mathcal{V}(x) \leq c$, where $\dot{\mathcal{V}}_{(B1)} < 0$. We have $\dot{\mathcal{V}}_{(B1)} = x^T Dx + e(x)$, where $e(x)$ is a cubic form, say, $e(x) = \Sigma_{i,j,k=1}^{5} e_{ijk} x_i x_j x_k$. Write e_x as a sum of terms of the form $x_i x_j \Sigma_{k=1}^{5} e_{ijk} x_k$. The combinations (i,j) are obviously not unique. For definiteness, let us select (1,5), (2,5), (3,5), (4,5), and (3,4) as (i,j). The set of the five inequalities

$$|\Sigma_{k=1}^{5} e_{ijk} x_k| / 2(1-\mu)(d_{ii} d_{jj})^{1/2} < 1 \qquad \text{(B2)}$$

with μ some small positive constant, say 10^{-3}, provides a sufficient condition for $\dot{\mathcal{V}}_{(B1)} < 0$. Compute the maximum value of c for which each of the five inequalities (B2) holds on the surface $\mathcal{V} = c$. Maximum on $\mathcal{V} = c$ of the left side of each inequality (B2) shall be calculated first. The problem is solved by the method of multipliers. Denote the algebraic value of the first member of an inequality (B2) by $f(x)$, and let λ be a scalar. Then, solving the system of equations in x and λ viz. $\partial f/\partial x + \lambda \partial \mathcal{V}/\partial x = 0$, $\mathcal{V}(x) = c$ yields the maximum on $\mathcal{V} = c$ of f. Now f is linear, $f = b^T x$, and the system can be written as $b + 2\lambda Vx = 0$, $x^T Vx = c$. From the first equation, we have $x = -(1/2\lambda) V^{-1} b$, and, accordingly, since V is symmetric, the second becomes $(1/4\lambda^2)\ b^T V^{-1}\ b = c$, whence $\frac{1}{2}\lambda = \pm c^{1/2}\ (b^T V^{-1} b)^{-1/2}$, $x = \mp c^{1/2}\ (b^T V^{-1} b)^{-1/2}\ V^{-1} b$, and $f(x) = \mp c^{1/2} (b^T V^{-1} b)^{1/2}$. The sign is plus, since we are seeking the maximum. From obvious symmetry considerations, it readily results that the maximum value of $|f(x)|$ on the surface $x^T Vx = c$ is again $c^{1/2} (b^T V^{-1} b)^{1/2}$. Hence inequality $c < (b^T V^{-1} b)^{-1}$ makes sure that this maximum is less than 1. Furthermore c is estimated in the same way for all inequlities (B2). Let $\bar{c}$ be the least estimate. Now, if

$$\mathcal{V}(x_0) < \bar{c} \qquad \text{(B3)}$$

then x_0 lies in the domain of attraction of the origin, and that means that the singular point $(\beta_s, \alpha_s, q_s, r_s, p_s)$ is an effective attractor.

Analysis is performed for aileron angles of 1.0, 1.2, 1.4, 1.6, 1.8, 2.0 deg for our numerical example. Only the stable singular points with negative p_s are selected, and the conclusions are summarized in Table 2.

Table 2 Conclusions of analysis

δ_a, deg	p_s, deg/s	$\bar{c}$	$\mathcal{V}(x_0)$	Does inequality ($B3$) hold?
1.00	-0.121×10^{2}	0.329	0.791×10^{-1}	Yes
	-0.266×10^{3}	0.130×10^{-1}	0.114×10^{4}	No
1.20	-0.145×10^{2}	0.332	0.115	Yes
	-0.266×10^{3}	0.146×10^{-1}	0.108×10^{4}	No
1.40	-0.169×10^{2}	0.335	0.158	Yes
	-0.266×10^{3}	0.162×10^{-1}	0.104×10^{4}	No
1.6	-0.193×10^{2}	0.338	0.209×10^{3}	Yes
	-0.266×10^{3}	0.178×10^{-1}	0.997×10^{3}	No
1.8	-0.217×10^{2}	0.342	0.268	Yes
	-0.266×10^{3}	0.199×10^{-1}	0.955×10^{3}	No
2.0	-0.242×10^{2}	0.345	0.336	Yes
	-0.266×10^{3}	0.219×10^{-1}	0.921×10^{3}	No

It is noteworthy that condition (B3) is sufficient without, however, being necessary. Hence it is conclusive only if one of the stable singular points fulfills it. Conversely, there might exist an effective attractor, or there might not if no singular point satisfies condition (B3). For this reason, the particular Lyapunov function or the criterion (B2) chosen previously no longer works satisfactorily for higher values of δ_a. A more suitable Lyapunov function should be constructed for larger aileron deflections, which would take account of the nonlinear terms of the system. As already mentioned, this would greatly enhance efficiency of the analysis. Unfortunately, however, this would hardly be an easy task. For there, too, as in most problems of practical interest, "the choice of an appropriate Lyapunov function is an art"[10] or simply a matter of good luck.

Acknowledgment

I wish to express my gratitude to C. Oprisiu, my collaborator and graduate student, for carrying out the machine calculations.

References

[1] Pinsker, W.J.G., "Preliminary Note on the Effect of Inertia Cross Coupling on Aircraft Response in Rolling Maneuvers," Aeronautical Research Council, C.P. 435, 1955.

[2] Thomas, H.H.B.M. and Price, P., "A Contribution to the Theory of Aircraft Response in Rolling Maneuvers Including Inertia Cross Coupling Effects," Aeronautical Research Council, R&M 3349, 1964.

[3] Hacker, T. and Oprisiu, C., "A Discussion of the Roll-Coupling Problem," *Progress in Aerospace Sciences,* Vol. 15, edited by D. Küchemann, Pergamon, New York, 1974, pp. 151-180.

[4] Rhoads, D.W. and Schuler, J.M., "A Theoretical and Experimental Study of Airplane Dynamics in Large-Disturbance Maneuvers," *Journal of the Aeronautical Sciences,* Vol. 24, July 1957, pp. 507-526, 532.

[5] Schy, A.A. and Hannah, M.E., "Prediction of Jump Phenomena in Roll-Coupled Maneuvers of Airplanes," *Journal of Aircraft,* Vol. 14, April 1977, pp. 375-382.

[6] Hacker, T., "Attitude Instability in Steady Rolling and Roll Resonance," *Journal of Aircraft,* Vol. 14, Jan. 1977, pp. 23-31.

[7] Hahn, W., *Stability of Motion,* Springer, New York, 1967, Sec. 26 and 33.

[8] Zubov, V.I., *Methods of A.M. Lyapunov and Their Application,* Office of Technical Services, Dept. of Commerce, Washington, D.C., AEC-TR-4439, Oct. 1961, Chap. 2.

[9] Ingwerson, D.R., "A Modified Lyapunov Method for Nonlinear Stability Analysis," *Institute of Radio Engineers Transactions,* Vol. AC-6, 1961, pp. 199-210.

[10] Halanay, A., "For and Against the Liapunov Function," *Symposia Mathematica,* Vol. 6, Instituto Nazionale di Alta Matematica, Monograf, Bologna, 1971, p. 167-175.